Total number of printed pages–24

3 (Sem–5/CBCS) MAT HE 1/HE 2/HE 3

2021

(Held in 2022)

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION–A

Paper : MAT–HE–5016

(Number Theory)

DSE (H)–1

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

PART–A

1. Choose the correct option : 1×10=10

   (i) Two integers a and b are coprime if there exists some integers x, y such that

   (a) $ax + by = 1$

Contd.

(b) $ax - by = 1$

(c) $(ax + by)^n = 1$

(d) None of the above

(ii) Let $d = gcd(a, b), n \in \mathbb{N}$. If $d \mid c$ and (x_0, y_0) is a solution of linear Diophantine equation $ax + by = c$, then all integral solutions are given by

(a) $(x, y) = \left(x_0 + \frac{bn}{d}, y_0 - \frac{an}{d}\right)$

(b) $(x, y) = \left(x_0 - \frac{bn}{d}, y_0 + \frac{an}{d}\right)$

(c) $(x, y) = \left(x_0 + \frac{an}{d}, y_0 - \frac{bn}{d}\right)$

(d) $(x, y) = \left(x_0 - \frac{an}{d}, y_0 + \frac{bn}{d}\right)$

(iii) A reduced residue system modulo m is a set of integers r_i such that

(a) $[r_i, m] = 1$

(b) $(r_i, m) = 1$

(c) $(r_i, m) \neq 1$

(d) None of the above

(iv) Suppose that m_j are pairwise relatively prime and a_j are arbitrary integers ($j = 1, 2, \ldots k$) then there exist solution x to the simultaneous congruence $x \equiv a_j (mod\ m_j)$, such that x are

(a) congruent modulo $M = m_1.m_2.m_3 \ldots m_k$

(b) congruent modulo $M = \sum_{j=1}^{k} m_j$

(c) congruent modulo m_i

(d) Both (a) and (b)

(v) The product of four consecutive positive integers is divisible by

(a) 20

(b) 22

(c) 24

(d) 26

(vi) Euler's ϕ-function of a prime number p, i.e., $\phi(p)$ is

(a) p

(b) $p - 1$

(c) $\frac{p}{2} - 1$

(d) None of the above

(vii) For which value of m,
CRS $(mod\ m)$ = RRS $(mod\ m)$?

(a) If m is a prime
(b) If m is a composite
(c) If $m < 10$
(d) None of the above

(viii) If $ca \equiv cb(mod\, m)$, then

(a) $a \equiv b\left(mod \dfrac{m}{(c, m)}\right)$
(b) $a \equiv b(mod\, m)$
(c) $a \equiv b(mod\, m.\ (c, m))$
(d) None of the above

(ix) The unit place digit of 2^{73} is

(a) 4
(b) 6
(c) 8
(d) 2

(x) The highest power of 7 that divides $50!$ is

(a) 7
(b) 8
(c) 10
(d) 5

2. Answer the following questions : $2 \times 5 = 10$

(a) If p is a prime, then prove that
$$\phi(p!) = (p-1)\,\phi((p-1)!)$$ 2

(b) Find all prime number p such that $p^2 + 2$ is also a prime. 2

(c) For $n = p^k$, p is a prime, prove that
$$n = \sum_{d|n} \phi(d)$$
where $\sum_{d|n}$ denotes the sum over all positive divisors of n. 2

(d) Find the number of zeros at the end of the product of first 100 natural numbers. 2

(e) Find $\sigma(12)$. 2

3. Answer ***any four*** questions : $5 \times 4 = 20$

(a) If ϕ is Euler's phi function, then find $\phi(\phi(1001))$. 5

(b) Find the remainder, when 30^{40} is divided by 17. 5

(c) State and prove Chinese Remainder Theorem. 5

(d) If p_n is the nth prime number, then prove that

$$p_n < 2^{2^{n-1}} \qquad 5$$

(e) If $n = p_1^{k_1}\, p_2^{k_2}\, p_3^{k_3} \ldots p_r^{k_r}$ is the prime factorization of $n > 1$, then prove that

(i) $\tau(n) = (k_1+1)(k_2+1)(k_3+1)\ldots(k_n+1)$

(ii) $\sigma(n) = \dfrac{p_1^{k_1+1}-1}{p_1-1} \times \dfrac{p_2^{k_2+1}-1}{p_2-1} \times \ldots \times \dfrac{p_r^{k_r+1}-1}{p_r-1}$

2½+2½=5

(f) Define Mobius function. Also show that

$$\mu(m \cdot n) = \mu(m) \cdot \mu(n)$$

Hence find $\mu(6)$. 1+3+1=5

PART–B

*Answer **any four** questions :* 10×4=40

4\. (a) If $d = (a, n)$, prove that the linear congruence $ax \equiv b \pmod{n}$ has a solution if and only if $d \mid b$. 5

(b) (i) When a number n is divided by 3 it leaves remainder 2. Find the remainder when $3n+6$ is divided by 3. 2

(ii) Prove that $5n+3$ and $7n+4$ are coprime to each other for any natural number n. 3

5\. (a) If p is a prime, then prove that

$$(p-1)! \equiv -1 \pmod{p} \qquad 5$$

(b) Using property of congruence show that 41 divides $2^{20}-1$. 5

6\. (a) Prove that every positive integer $(n > 1)$ can be expressed uniquely as a product of primes. 5

(b) Determine all solutions in the integers of the Diophantine equation

$$172x + 20y = 1000 \qquad 5$$

7\. (a) If n be any positive integer and can be expressed as $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$, then prove that $\phi(n) = n\prod_{j=1}^{k}\left(1-\dfrac{1}{p_j}\right)$. 5

(b) If m and n are any two integers such that $(m, n) = 1$, prove that

$$\phi(m.n) = \phi(m).\,\phi(n).$$ 5

8. (a) For each positive integer $n \geq 1$, show that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$ 5

(b) If k denotes the number of distinct prime factors of positive integer n, then prove that

$$\sum_{d|n} |\mu(d)| = 2^k$$ 5

9. (a) Show that $\sum_{d|n} \mu(d)\,\tau(d) = (-1)^k$

where k denotes the number of distinct prime factors of positive integers n. 5

(b) Prove that

(i) $\tau(n)$ is an odd integer iff n is a perfect square. 3

(ii) For any integer $n \geq 3$, show that

$$\sum_{k=1}^{n} \mu(k!) = 1.$$ 2

10. (a) Let p be an odd prime. Show that the congruence $x^2 \equiv -1 (mod\, p)$ has a solution if and only if $p \equiv 1 (mod\, 4)$. 5

(b) If $n \geq 1$ and $gcd(a, n) = 1$, then prove that $a^{\phi(n)} \equiv 1 (mod\, n)$. 5

11. (a) If n is a positive integer and p is a prime, then prove that the exponent of the highest power of p that divides $n!$ is $\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$. 5

(b) Solve $3[x] = x + 2\{x\}$ where $[x]$ denotes greatest integer $\leq x$ and $\{x\}$ denotes the fractional part of x. 5
